I. THE TERM STRUCTURE OF GOVERNMENT BOND YIELDS IN AN EMERGING MARKET

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Abstract

The accurate modeling of the term structure of interest rates is of vital importance in macroeconomics and finance in general and in the context of monetary policy in particular, as its factors are important in predicting future growth and inflation. This paper investigates the extent to which the so called Nelson-Siegel model (DNS) and its extended version that accounts for time varying volatility (DNS-GARCH and DNS-EGARCH) can optimally fit the yield curve and predict its future path in the context of an emerging economy. The study expands the earlier work (Koopman, et al. 2010) by looking at more elaborate specifications for volatility modeling such as E-GARCH and also evaluates the predictive role of considering the time-varying volatility in the model in terms of out-of-sample forecasting. For the in-sample fit, all three models fit the curve remarkably well even in the emerging markets. However, the DNS-EGARCH model fits the curve slightly better than the other two models. Moreover, all three specifications of the yield curve that are based on the Nelson-Siegel functional form, outperform the benchmark AR(1) forecasts at all three specified forecast horizons. The DNS comes with more precise forecasts than the volatility based extended models for the 1-month ahead forecasts, while the other two outperform the standard DNS for 6- and 12-month horizons.

Keywords: yield curve, forecasting, emerging markets, Kalman filter, EGARCH
JEL Classification: C32, C53, C51, E43, G12, G17

1. Introduction

The yield curve describes the relationship between yields and maturity on zero-coupon bonds that are homogeneous in every aspect except time to maturity. These yields are the set of interest rates derived from equating the current market price of government bonds to the discounted stream of future cash flows of the bond. Since, the term structure can be formed by using the prices of zero coupon bonds. However, because of the limited maturity spectrum and lack of market liquidity of the zero-coupon bonds, it is essential to estimate

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the yield based on observed coupon bond prices. Furthermore, the yield curve holds information about the market's expectations of future events. Therefore, interest rate forecasts can be used to predict this information for identiying profitable investment opportunities and as a general guideline for economic policy.

The literature on term structure goes back to Hicks (1939) and Lutz (1940) discussion of expectation hypothesis, implying that the current yield curve contains information about the future path of interest rates. According to the expectation hypothesis, the current forward rates will act in such way that exactly follows the path of future short-term zero-coupon rates. A model that forms the basis of many other term structure models considering uncertainty in the market is the Vasicek (1977) model. Other famous models of this nature include Brennan and Schwartz (1979) and Cox *et al.* (1985) that rely on continuous-time finance theory and no arbitrage restriction to estimate and forecast the term structure, while allowing for multiple sources of uncertainty. Among others, Ang and Piazzesi (2003), Favero *et al.* (2012) and Mönch (2008) applied no-arbitrage restrictions to VAR models, Diebold and Li (2006), de Pooter (2007) and Christensen *et al.* (2011) examine dynamic versions (with and without no-arbitrage restrictions) of the Nelson–Siegel model of the cross section of yields.

Other recent comparative forecasting exercises include Almeida and Vincente (2008), Carriero (2011), Duffee (2011), Carriero and Giacomini (2011), Yu and Zivot (2011) and Ullah *et al.* (2013). The studies are mixed in their conclusions and also in whether adding no-arbitrage restrictions to dynamic models of yields adds value to forecasts. Despite these controversies, more positive results have been obtained by employing the Nelson and Siegel (1987) framework that imposes a parsimonious three-factor structure to link the yields of different maturities, where the factors can be interpreted as level, slope and curvature. Although the model is not based on any underlying economic theory and is of statistical nature, it is still widely used due to its good fit of the observed term structure and generation of forecasts that outperform the random walk and various alternative forecasting approaches.

Despite recent advances in forecasting literature, there has been little evidence supporting the usefulness of these models to forecast yields in emerging markets. The reasons are (i) the lack of good quality data, and (ii) limited time span data, which makes it very difficult to reach sound conclusions. A salient trait of this strand of literature is the strong emphasis it places on the US economy and international evidence has remained scarce and limited. Furthermore, most of the evidences regarding the Nelson-Siegel family of models are based on pricing data obtained from developed economies bond markets such as USA (Diebold and Li, 2006; Christensen *et al.* 2011 and many others), Euro area (Silvana and D'Ecclesiab, 2008; Steeley, 2014) and Japan (Kim and Singleton, 2012; Kikuchi and Shintani, 2012; Ullah *et al.* 2013, 2014), where the markets are efficient and informational contents are reflected fully and instantaneously in the prices. There are only few studies that evaluate the performance of the standard Nelson-Siegel model and its extended versions or other statistical models of term structure in terms of in-sample fit and out-of-sample forecast in the context of emerging markets.

Moreover, the bond market in Pakistan has expanded rapidly over the last 10 years, as it is currently the main source of financing for the government. The central bank is also trying to promote the development of the bond market by generating new products and the establishment of a group of primary dealers responsible for stimulating the market. In spite of this more than 90% of the issued bonds are government securities and the number of corporate issuers remains limited. Furthermore, the bond market is marked by a significantly lower volume of trading as it is smaller and illiquid compared to bond markets of developed countries. In fact, the sparseness or infrequency of daily Treasury bonds transactions explains in consequence the inaccuracy of the interest rates yield curve. The drawback is that there is no specific term structure model of interest rates and market operators devise a proxy of the yield curve based only on the liquid bonds. Thus, the unevenly distributed maturities of different bonds makes the estimation very difficult and the market less likely to form an entire and smooth yield curve. This study attempts to fill this gap by investigating the Nelson-Siegel model and its extensions in terms of their ability to find a smooth yield curve which replicates the stylized facts of the various interest rates in the context of the Pakistan bond market.

The focus of this study is to identify the most appropriate model for predicting the term structure of interest rates in the context of emerging markets. To this end, we formulate the standard Nelson-Siegel model (DNS) and its extended forms that account for the time varying volatility in the bond market (DNS-GARCH and DNS-EGARCH) in the state space framework to determine which of these three term structure models are suited for fitting and forecasting purposes.

Against this background, this paper makes three contributions to existing literature. First, it investigates whether the promising results of the Nelson-Siegel model can be generalized to the bond markets of emerging economies, where the markets suffer from lack of liquidity and governments rely heavily on bond financing to finance its deficits. Secondly, this paper is the first attempt to investigate in a systematic and comparative fashion the Nelson-Siegel family of yield curve models in the government bond market of the Pakistan economy. Thirdly, we show that the inclusion of EGARCH effect is not only helpful in terms of in-sample fitting but also for long horizon forecasts.

The remainder of the paper is structured as follows. Section 2 briefly describes the dynamic Nelson-Siegel model and its extended versions that account for time-varying volatility (we call the former DNS and the latter as DNS-GARCH or DNS-EGARCH). Estimation method is also discussed in the same section. Section 3 presents the data structure and estimation results, while section 4 describes the out-of-sample forecast performance of the models. Finally, section 5 concludes the paper.

2. Literature Review

The literature on term structure modeling can be classified in different ways. One strand of literature uses models that impose the restriction of no arbitrage on the evolution of yields to avoid riskless opportunities, known as the affine class of models. Arbitrage-free models arise from equilibrium models and, therefore are based on sound economic foundation. Seminal works in this regard include Vasicek (1977), Cox *et al.* (1985) and Heath *et al.* (1992). Despite having the appealing characteristic of no-arbitrage restriction, the arbitrage –free class suffers from lack of fitting and predicting the yield curve (Duffee, 2002). The other class consists of purely statistical models. The pioneer works in the class of statistical models include Nelson and Siegel (1987) and Svensson (1995). Although the original Nelson-Siegel equation gives a good shape of the yield curve for the selected data, it is still not able to solve problems involving complex data sets. Later, the original Nelson-Siegel method was improved by adding additional factors in the model to estimate and forecast the yield curve. By including the additional curvature factor as in Svensson, (1995) and second slope as in Bliss (1996), the extended Nelson-Siegel model with four factors produces structural accuracy with longer maturities. Among others, Diebold and Li (2006) propose a two-stage model based on the Nelson and Siegel (1987) framework to forecast the US term structure that presented better results than other competing models. Their results beat the random walk, with the exception of the 1-month ahead horizon.

After Diebold and Li (2006), many authors have developed different dynamic approaches of the Nelson–Siegel model and have reported improvements in forecasting. Koopman et al. (2010) introduce time-varying volatility in the DNS framework and conclude that the models with time-varying volatility outperform the models with static volatility in terms of in-sample fit for the US bond market. Similarly, Ullah (2017) shows that considering the time-varying volatility in the Nelson-Siegel model not only yields better in-sample fit but also improves the out-of-sample accuracy in the Japanese market. Diebold et al. (2008) extend the model to a global context, modeling a large set of country yield curves in a framework that allows for both global and country-specific factors.

The implementation of yield curve modeling and forecasting in the context of emerging economies is recent. In the case of the Brazilian bond market yield curve, Vicente and Tabak (2008) show that the dynamic Nelson-Siegel model dominates forecasts made from affine models and is also better than random walk forecasts for some maturities and forecast horizons. Similar results are also presented in Cajueiro et al. (2009). The studies on the estimation of the term structure of interest rates in the Indian market show that cubic spline based methods have larger errors compared to the Nelson-Siegel and Svensson (1995) type models (Dutta, *et al.* 2005). The cubic B-spline and cubic spline with violence or smoothing spline penalty methods did not achieve the objectives for curve estimation. In the context of Taiwan and Malaysian bond markets (with small size of bond trading and lower liquidity level), Chou, *et al.* (2009) and Ali, *et al.* (2015) respectively suggest that the Nelson-Siegel model or its extended versions are capable of describing the shape of the term structure and forecast the term structure more accurately. Moreover, Chou, et al. (2009) show that the fitting performance of Svensson model is better than that of Bliss and Nelson-Siegel Model taking into account the liquidity constraint. They also compared the results with the case in which the liquidity constraint is not taken into consideration, these three models have a better fitting performance if the liquidity constraint is considered. This suggests that the liquidity constraint matters. In addition, Araujo and Cajueiro (2013) and Caldeira *et al.* (2016) show that it is not possible to determine an individual model that consistently produces superior forecasts for all maturities and all forecast horizons. Nevertheless, empirical results suggest that the traditional DNS model has good out-of-sample forecasting performance when compared to the RW, AR(1), and VAR(1), especially when we consider 1- and 3-month ahead forecast horizons.

Overall, the results show that there is no single forecast model that dominates all competitors. This is due to the fact that different models outperform the others, depending on time horizon ahead, maturity and forecast period.

3. Term Structure Models

The term structure of interest rates refers to the relationship between interest rates and time to maturity. The standard way to compute the term structure of interest rates is to plot the zero rates (derived from zero-coupon bonds) against the entire maturity spectrum. However, the limited maturity spectrum of zero-coupon bonds necessitates that the yields be derived from the coupon bearing bonds by considering each strip (coupon payment) as a distinct zero-coupon bond. Moreover, due to lack of market liquidity of long maturity bonds one cannot compute the zero rates for the entire maturity spectrum. Therefore, some sort of model is required to fill the gaps by analogy with the observed rates. In this section, we briefly describe the dynamic Nelson-Siegel (DNS) model for the computation of yield curve and its two extended versions that account for the common volatility component (modeled as GARCH and an EGARCH process). For the ease of writing and interpretation, we termed the standard dynamic Nelson-Siegel model as DNS, and the two extended models as DNS-GARCH (where the variance of common volatility component is modeled as GARCH) and DNS-EGARCH (where the variance of common volatility component is modeled as an EGARCH process).

3.1. The Nelson-Siegel Model

Motivated by the expectations hypothesis, Nelson-Siegel (1987) introduced a three-factor model that is capable of explaining about 96% of the variation of the yield curve across maturities. The expectation hypothesis states that the rationally expected future spot rates will be equal to the current implied forward rates, i.e., forward rates will fluctuate in such a way that guarantees no arbitrage opportunity in the market. Nelson and Siegel (1987) suggest that the spot rates curve can be generated with the help of differential or difference equation. If the differential equation implies the spot rates, then implied forward rates will be the solution to this equation. Supposing that spot rates are constructed with the help of second order differential equation, then the functional form for the instantaneous implied forward rate can be written as:

$$f_t(m) = \beta_{1t} + \beta_{2t}\exp(-\lambda m) + \beta_{3t}[(\lambda m)\exp(-\lambda m)] \quad (1)$$

with the time varying parameter vector $\beta_t = (\beta_{1t}, \beta_{2t}, \beta_{3t})'$. The $f_t(m)$ is the instantenous forward rate, m is time to maturity and $\lambda = 1/\tau$, where τ is the real root of the differential equation. The resulting yield curve function, computed as $R_t(m) = m^{-1} \int_0^m f_t(u)du$, can be defined as:

$$R_t(m) = \beta_{1t} + \beta_{2t}\left[\frac{1-\exp(-\lambda m)}{\lambda m}\right] + \beta_{3t}\left[\frac{1-\exp(-\lambda m)}{\lambda m} - \exp(-\lambda m)\right] + \varepsilon_t(m) \quad (2)$$

where: $R_t(m)$ is the zero-coupon spot rate for maturity $m = 1,2,\dots,N$ at $t = 1,2,\dots,T$. The functional form in (2) can produce various forms of curves including upward, downward, and humped shapes that are usually associated with the term structure of interest rate. Moreover, the parameters in Nelson-Siegel functional form can easily be interpreted. The limiting value of spot rate curve as $m \to 0$ is $\beta_{1t} + \beta_{2t}$, and when $m \to \infty$ the resulting value is β_{1t}. Since, β_{1t} can be interpreted as the long term interest rate, while $-\beta_{2t}$ as term premium or slope of the yield curve.

Dielold and Li (2006) show that β_{1t}, β_{2t} and β_{3t} can be interpreted as the yield curve latent factors. The loading of β_{1t} is unitary across all maturities and does not approach zero in the limit and, therefore, can be considered as the level factor. The β_{2t} serves as the spread factor and can be interpreted as the slope of the yield curve, while β_{3t} mainly affects the middle part of the curve and, hence, can be considered the curvature factor of the curve. The loading of β_{3t} at both ends is zero but it reaches its maximum point at some intermediate maturity, in the range of 30 to 60 months in empirical literature. Lastly, the parameter λ specifies the maturity time at which the loading of the curvature factor β_{3t} is optimal and also identifies the location of the U or the hump-shape on the yield curve. Therefore, the variety of shapes the curve can take is dependent on a single parameter λ, which represents the rate at which the slope and curvature factor loadings decay to zero.

The formulation of the dynamic Nelson-Siegel (DNS) model is parsimonious and simple to be estimated. However, to model all yield curves in a single step, this calls for a state-space representation of the model.[3] Since, we assume the first order vector autoregressive representation for the yield curve latent factors vector β_t, which facilitates the state space representation of the latent factors model, with measurement and state equations (3 and 4 respectively) as:

$$R_t = \Lambda(\lambda)\beta_t + \varepsilon_t \quad (3)$$

$$\beta_{t+1} = (I_3 - A)\mu + A\beta_t + v_{t+1} \quad (4)$$

$$\begin{bmatrix} \varepsilon_t \\ v_{t+1} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & \Sigma_v \end{bmatrix}\right) \quad (5)$$

where: R_t is (N×1) vector of zero-coupon yields, $\beta_t = (\beta_{1t}, \beta_{2t}, \beta_{3t})'$ is the (3×1) vector of latent factors of the yield curve, μ is (3×1) vector of factors mean, $\Lambda(\lambda)$ is (N×3) matrix of loadings and A is (3×3) full-matrix of parameters. The ε_t and v_t are (N×1) and (3×1) errors vectors of the observation and state equations respectively, Ω is (N×N) covariance matrix of the measurement equation innovations, and Σ_v is (3×3) covariance matrix of the state innovations.

3.2. The Dynamic Nelson-Siegel Model with Time-varying Volatility (DNS-GARCH and DNS-EGARCH)

In the standard DNS model, it is assumed that the volatility in interest rates is time invariant. However, interest rates are the outcomes of trading in financial markets and, therefore, the volatility in various yields may also vary across time. Hence, we add a common volatility component to the yield curve specification while assuming the state space approach. This addition enables the standard DNS model to apprehend the exogenous shocks, which shift the yield curve and are not taken into account by the level, slope and curvature factors. The error term, ε_t, in the DNS model is restructured as:

$$\varepsilon_t = \Gamma_\varepsilon \varepsilon_t^* + \varepsilon_t^+ , \qquad \varepsilon_t^+ \sim N(0, \Omega) \quad (6)$$

where: ε_t^+ and Γ_ε are (N×1) vectors of noise component and loadings respectively, and ε_t^* is a scalar representing the common shock term. The loading factor, Γ_ε, shows the sensitivity of various yields to the common disturbance term. The conditional distribution of the common volatility component, ε_t^*, is assumed as:

$$\varepsilon_t^* | \zeta_{t-1} \sim N(0, h_t) \quad (7)$$

where: ζ_{t-1} is the given information set up to time $t-1$ and h_t follows the standard GARCH specification (in DNS-GARCH model) and EGARCH (in DNS-EGARCH model) specification, which is specified in (8) for the DNS-GARCH and (9) for the DNS-EGARCH based model.[4]

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[3] *Moreover, Diebold* et al. *(2006) show that the three latent factors in the Nelson-Siegel model are highly persistent. This means that we can model them as AR(1) or VAR(1). Similarly, Ullah* et al. *(2013) used the Japanese data and confirmed that the three latent factors of yield curve are highly persistent. They found VAR(1) specification to be better than the AR(1) and random walk specifications.*

[4] *The response of financial market differs with the nature of the shock. It is an accepted fact that volatility increases rapidly with negative news reaching the traders and investors. The impact of similar magnitude positive news tends to be much less pronounced.*

$$\log(h_t) = \gamma_0 + \gamma_1 \frac{\varepsilon^*_{t-1}}{\sqrt{h_{t-1}}} + \gamma_2 \log(h_{t-1}) \tag{8}$$

$$\log(h_t) = \gamma_0 + \gamma_1 \frac{\varepsilon^*_{t-1}}{\sqrt{h_{t-1}}} + \gamma_2 \log(h_{t-1}) + \psi \left(\left| \frac{\varepsilon^*_{t-1}}{\sqrt{h_{t-1}}} \right| - \mathbb{E}\left[\left| \frac{\varepsilon^*_{t-1}}{\sqrt{h_{t-1}}} \right| \right] \right) \tag{9}$$

where: $\mathbb{E}\left(\left|\varepsilon^*_{t-1}/\sqrt{h_{t-1}}\right|\right)$ is the expectation of the absolute value of a standard normally distributed random variable, which is equal to $\sqrt{2/\pi}$. The specification for variance in (8) assumes that that the volatility in various yields is governed by the latent exogenous shocks and lagged volatility, whereas in (9), besides the exogenous shocks and lagged volatility, we take into account for the asymmetric response to positive (good) and negative (bad) shocks.

In the state-space framework the DNS-GARCH and DNS-EGARCH can be specified as:

$$R_t = [\Lambda(\lambda) \quad \Gamma_\varepsilon] \begin{bmatrix} \beta_t \\ \varepsilon^*_t \end{bmatrix} + \varepsilon^+_t \tag{10}$$

$$\alpha_{t+1} = \begin{bmatrix} (I_3 - A)\mu \\ 0 \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \alpha_t + \begin{bmatrix} v_{t+1} \\ \varepsilon^*_{t+1} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \varepsilon^+_t \\ v_{t+1} \\ \varepsilon^*_{t+1} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega & 0 & 0 \\ 0 & \Sigma_v & 0 \\ 0 & 0 & h_{t+1} \end{bmatrix} \right) \tag{12}$$

where: $\alpha_t = (\beta_{1t}, \beta_{2t}, \beta_{3t}, \varepsilon^*_t)'$ is (4×1) latent vector. Γ_ε (N×1) vector shows the impact of a common shock component on the various yields. The remaining matrices and vectors have the same definitions and dimensions as discussed in the specification of the DNS model. We further assume that the innovations, ε^+_t and v_t, and common volatility component, ε^*_t, have Gaussian distribution. The variance of ε^*_{t+1} is h_{t+1} is modeled as GARCH or EGARCH processes, specified in (8) and (9) respectively. The model in equations (8 – 12) stipulates a comprehensive and flexible framework to fit the yield curve. It also accounts for the time-varying volatility in yields for all maturities.

3.3. Statistical Formulation of the Models and Estimation Method

The models are estimated with the Kalman filter algorithm. To explain the estimation procedure for all three frameworks in a comprehensive way, we introduce a generalized framework. This framework uses some new notations. Signal and state equations are rewritten as:

$$R_t = B\xi_t + w_t, \quad \forall\, t = 1,2,\dots,T \tag{13}$$

$$\xi_t = C + F\xi_{t-1} + u_t \tag{14}$$

$$\begin{bmatrix} w_t \\ u_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega & 0 \\ 0 & Q_t \end{bmatrix} \right) \tag{15}$$

where: the expressions of $B, \xi_t, C, F, \Omega, Q_t, w_t$ and u_t for all three frameworks are given in Appendix-I.

The Kalman filter algorithm is applied as discussed in Hamilton (1994). The Gaussian likelihood function is evaluated to estimate the latent factors and the parameters. The Kalman filter iteration is initialized at the unconditional mean and variance of the state variables. The optimal estimate of the latent factors in Kalman filter is the conditional mean of ξ_t dependent on information known until time $t-1$ or t, denoted as $\hat{\xi}_{t|t-1}$ and $\hat{\xi}_{t|t}$ respectively. We calculate the recursive prediction step using the transition equation:

$$\hat{\xi}_{t|t-1} = \mathbb{E}_{t-1}(\xi_t) = C + F\hat{\xi}_{t-1|t-1} \tag{16}$$

$$P_{t|t-1} = \mathbb{E}_{t-1}\left[\left(\xi_t - \hat{\xi}_{t|t-1}\right)\left(\xi_t - \hat{\xi}_{t|t-1}\right)'\right] = FP_{t-1}F' + Q_t \tag{17}$$

where: $P_{t|t-1}$ is the mean square error (MSE) matrix at the prediction step and $Q_t = Q$ for the simple DNS model.
Using the observation equation, the prediction step estimates are updated by observing R_t , thus in the update step:

$$\hat{\xi}_{t|t} = \mathbb{E}_t(\xi_t) = \hat{\xi}_{t|t-1} + P_{t|t-1}B'H_t^{-1}\eta_t \tag{18}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}B'H_t^{-1}BP_{t|t-1} \tag{19}$$

where: η_t is the forecast errors vector calculated as: $\eta_t = R_t - B\hat{\xi}_{t|t-1}$ and H_t is the MSE matrix of η_t worked out as: $H_t = BP_{t|t-1}B' + \Omega$.
The Kalman filter iterative process is initialized with ξ_0 and P_0 that are set equal to unconditional mean and covariance as discussed in Hamilton (1994). The last diagonal element of P_0 in GARCH and EGARCH based models is set equal to h_1, which is the unconditional expectation of the log variance.
Furthermore, in the time-varying volatility based models, matrix Q_t contains h_{t+1} that is modeled by (E)GARCH process and relies on latent shocks at time t, which are unobservable. The h_{t+1} is computed by taking the conditional expectation at $t-1$ of the latent variables in (8) and (9) for the DNS-GARCH (specified in 20) and DNS-EGARCH (specified in 21) respectively that give:

$$\log(h_t) = \gamma_0 + \gamma_1\mathbb{E}_{t-1}\left(\frac{\varepsilon_{t-1}^*}{\sqrt{h_{t-1}}}\right) + \gamma_2\log(h_{t-1}) \tag{20}$$

$$\begin{aligned}\log(h_t) = \gamma_0 + \gamma_1\mathbb{E}_{t-1}\left(\frac{\varepsilon_{t-1}^*}{\sqrt{h_{t-1}}}\right) + \gamma_2\log(h_{t-1})\\ + \psi\mathbb{E}_{t-1}\left(\left|\frac{\varepsilon_{t-1}^*}{\sqrt{h_{t-1}}}\right| - \mathbb{E}_{t-1}\left[\left|\frac{\varepsilon_{t-1}^*}{\sqrt{h_{t-1}}}\right|\right]\right)\end{aligned} \tag{21}$$

where: the estimate of $\mathbb{E}_{t-1}(\varepsilon_{t-1}^*)$ is the last element of $\hat{\xi}_{t|t}$ from the update step. The beginning of the Kalman filter iteration depends on initial state ξ_0, initial covariance matrix P_0 and parameters vector θ. Defining $\theta = (\lambda, \mathrm{B}, C, F, \Omega, \mathrm{Q}_t, \Gamma_\varepsilon, \gamma_0, \gamma_1, \gamma_2, \psi)$ as the unknown parameters vector, and assuming a Gaussian distribution for the forecasting errors η_t, the Gaussian log likelihood is computed as:

$$\log L\,(\theta) = \sum_{t=1}^{T}\left(-\frac{N}{2}\log(2\pi) - \frac{1}{2}\log|H_t| - \frac{1}{2}\eta_t'H_t^{-1}\eta_t\right) \tag{22}$$

The Matlab based numerical optimization routine of fminsearch is employed to optimize the log likelihood function (22) and obtain the estimates of the parameters.

4. Empirical Results

The empirical results regarding the in-sample fitting performance of the three models, i.e., DNS, DNS-GARCH and DNS-EGARCH are presented in this section. Here, we answer questions that does considering the common volatility component in the term structure model enhance the performance of the underlying model and what are the underlying factors in deriving the yields for various maturities? The Kalman filter algorithm is employed to the zero-coupon yields data for various maturities in the bond market of Pakistan to attain the optimal estimates of the latent factors and the MLE estimates of the unknown parameters. Sections 3.1 and 3.2 give details of the data-set and estimation results respectively.

4.1. Data and Summary Statistics

We use the Pakistan yield data published by the Mutual Fund Association of Pakistan (MUFAP) and Pak Brunei Investment Company. We collect the monthly observations for the period from August 2002 until December 2016 on yields for 15 maturities of 3, 6, 9, 12, 18, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months.

The descriptive statistics for the yields are presented in Table 1. The results show the average yield curve to be upward sloping as the mean yield increases with maturity. Moreover, the short rates are found to be more volatile and persistent than long rates. Skewness exhibits an upward trend with maturity. Kurtosis of the short rates are lower than those of the long rates. The yields for all maturities are also highly persistent.

Table 1

Descriptive statistics of yields data across maturities

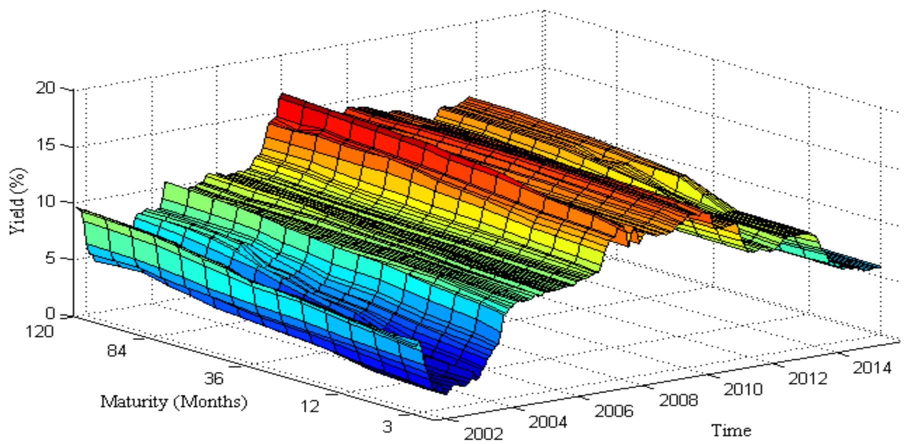
Maturity	Mean	SD	Max	Min	SK	Kurtosis	$\hat{\rho}(1)$	$\hat{\rho}(6)$	$\hat{\rho}(12)$
3	8.552	3.426	13.449	1.118	-0.561	2.427	0.990	0.890	0.703
6	8.660	3.436	13.751	1.131	-0.533	2.439	0.990	0.887	0.694
9	8.744	3.411	13.914	1.233	-0.522	2.445	0.990	0.883	0.686
12	8.828	3.388	14.109	1.335	-0.511	2.449	0.990	0.880	0.678
18	9.054	3.309	14.570	1.726	-0.489	2.381	0.989	0.867	0.663
24	9.279	3.245	15.063	2.117	-0.456	2.313	0.987	0.848	0.642
30	9.424	3.205	15.271	2.316	-0.441	2.272	0.987	0.844	0.635
36	9.569	3.171	15.478	2.515	-0.422	2.232	0.986	0.838	0.625
48	9.831	3.036	15.841	3.096	-0.397	2.276	0.985	0.825	0.618
60	10.010	2.956	15.866	3.474	-0.381	2.275	0.984	0.816	0.605
72	10.236	2.803	16.087	3.845	-0.375	2.391	0.983	0.798	0.583
84	10.389	2.711	16.183	4.192	-0.356	2.409	0.982	0.792	0.579
96	10.512	2.621	16.266	4.584	-0.309	2.350	0.981	0.790	0.580
108	10.572	2.598	16.389	4.683	-0.326	2.392	0.981	0.790	0.585
120	10.628	2.604	16.531	4.531	-0.345	2.528	0.980	0.782	0.570

Note: The table shows descriptive statistics for monthly yields at different maturities. The last three columns contain sample autocorrelations at displacements of 1, 6 and 12 months. The sample period is 2002:08–2016:12. The number of observations is 173.

Figure 1 presents a three-dimensional plot of the yield curve data. The visual inspection indicates that the yield curves have an upward slope at all points of time considered in this study. Moreover, the shape is almost stable except early 2006 and 2010. The figure shows that the yield curves have shifted down in the current episode of monetary policy ranging from early 2015 till date. This phenomenon is also reflected in the estimated conditional volatility for the DNS-EGARCH model in Figure 2. These statistics provide the first evidence of a change in the dynamics of the yield curve as a result of the rise in interest rates in the Pakistan bond market.

Figure 1

Yield Curves - Data Plot



The figure shows the yield curves, 2002:08–2016:12. The sample consists of monthly yield data from August 2002 to December 2016 (173 months) for maturities of 3, 6, 9, 12, 18, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months (15 maturities).

4.2. Estimation Results

In Table 2, we present the estimates of mean vector μ and transition matrix A for all three specifications of dynamic Nelson-Siegel model, *i.e.,* DNS, DNS-GARCH and DNS-EGARCH. The mean vector is statistically significant in all setups, however, the estimates of the elements of μ for the level and slope factors in the DNS-EGARCH are a bit larger as compared to the one in DNS-GARCH and DNS frameworks, whereas the estimated mean for curvature is larger in the latter two setups than the DNS-EGARCH model.

The diagonal elements of matrix A, which represent the own lag dynamics of factors, are highly significant and indicate that the yield curve factors are highly persistent. However, the level and curvature factors in time-varying volatility based frameworks are a bit more persistent than their counterpart DNS setup, while the slope factor seems more persistent in the DNS model than the rest of the two models. Cross factors dynamics seem unimportant except for the $\beta_{1,t-1}$ impact on the curvature and slope factors in the DNS and DNS-GARCH models respectively. In the EGARCH based model, there is significant impact of the level factor on both slope and curvature factors and lagged curvature on the level factor. The significant cross factors effect inspires for the VAR specification rather than the more parsimonious AR specification for modeling the yield curve factors. Moreover, the estimates of the decay parameter λ in all three setups are significant and imply that the inflection points occur at about 46 months' 42 months' and 48 months' maturity in DNS-EGARCH, DNS-GARCH and DNS models respectively. The variation in optimality point of the curvature factor loading may be due to the additional parameters in the volatility based setups.

Table 2

Latent Factors and (E)GARCH Parameter Estimates of the DNS-EGARCH, DNS-GARCH and DNS Models

Panel 1: Estimates of matrix $\boldsymbol{A}$ and vector $\boldsymbol{\mu}$				
	μ	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$
DNS-EGARCH				
$\beta_{1,t}$	9.805 (0.358)	0.862 (0.009)	0.021 (0.097)	0.213 (0.146)
$\beta_{2,t}$	-6.020 (0.206)	0.411 (0.004)	0.924 (0.022)	0.348 (0.043)
$\beta_{3,t}$	-2.021 (0.192)	0.007 (0.004)	0.015 (0.012)	0.828 (0.033)
λ	0.03911 (0.001)		Log L	1419.0281
DNS-GARCH				
$\beta_{1,t}$	9.761 (0.827)	0.834 (0.192)	0.033 (0.188)	0.163 (0.139)
$\beta_{2,t}$	-6.993 (0.541)	0.298 (0.117)	0.799 (0.148)	0.320 (0.053)
$\beta_{3,t}$	-1.072 (0.241)	-0.046 (0.033)	0.036 (0.026)	0.787 (0.342)
λ	0.043 (0.003)		Log L	1449.485
DNS				
$\beta_{1,t}$	9.672 (0.090)	0.6239 (0.0046)	0.034 (0.020)	0.179 (0.013)
$\beta_{2,t}$	-7.877 (0.362)	0.0821 (0.0278)	0.967 (0.015)	0.271 (0.032)
$\beta_{3,t}$	-1.866 (0.526)	-0.1451 (0.179)	0.031 (0.021)	0.729 (0.036)
λ	0.038 (0.002)		Log L	1372.569
Panel 2: (E)GARCH model parameter estimates in the DNS-EGARCH and DNS-GARCH models				
	γ_0	γ_1	γ_2	ψ
DNS-EGARCH	0.339 (0.001)	-0.089 (0.001)	0.628 (0.002)	-0.608 (0.001)
DNS-GARCH	0.332 (0.024)	0.277 (0.003)	0.596 (0.072)	-

Note: The table reports the estimates for the parameters of the transition equation of the simple DNS, DNS-GARCH and DNS-EGARCH yields models and of (E)GARCH parameters estimates in the DNS-GARCH and DNS-EGARACH models. Panel 1 presents the estimates for the vector μ and matrix A along with the decay parameter λ estimate, while panel 2 shows the parameters' estimates of the volatility processes (GARCH and EGARCH) of the common component in the DNS-GARCH and DNS-EGARCH based models. The standard errors are in parenthesis. Bold entries denote that parameter estimates are significant at the 5% level.

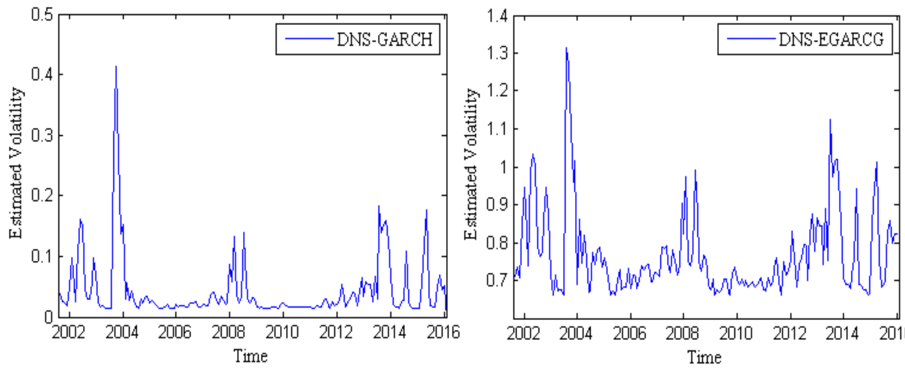
Regarding the volatility process parameters estimates in GARCH and EGARCH based models, the results are reported in panel 2 of Table 2. All estimates are statistically significant and also point out that lagged volatility plays a more dominant role than exogenous shocks in explaining current volatility in both setups. Furthermore, the bond market reacts asymmetrically to positive and negative shocks, as the estimate of asymmetry parameter ψ is negative and highly significant. The statistical significance of asymmetric effect parameter ψ and the residuals diagnostic of DNS model (skewness and excess kurtosis) motivates modelling volatility as an EGARCH rather than GARCH process. The residuals of DNS model are characterized by fat tails, as the excess kurtosis is positive for most of the maturities. The skewness is negative highlighting the asymmetry to the left of the residuals. Both stylized facts are rather stable for most of the maturities.

To illustrate more clearly the pattern of common volatility in the bond market, in Figure 2, the conditional volatility (h_t) of both GARCH and EGARCH based models is plotted over time. At first glance it seems that the conditional volatility process follows the same pattern across both models, however, the estimated volatility is higher at every point of time in

DNS-EGARCH than the DNS-GARCH model. It may be due to the additional term that captures the asymmetric effect and the comparatively weak impact of exogenous shocks effect (as $\gamma_1$coefficient is very small) in the EGARCH based framework as compared to the GRACH based setup.

Figure 2

DNS-GARCH and DNS-EGARCH Common Volatility (h_t)



The figure shows the plot of the volatility (h_t) of the common shock component (ε_t^*), which is modelled as GARCH process in the DNS-GARCH, while as an EGARCH in DNS-EGARCH model, over time.

Some historical events are clearly reflected in the graph. The last two big jumps correspond to the monetary and fiscal policy regimes in Pakistan. It shows that the yield curve responds to monetary policy stances and transmits the signals of monetary interventions to the real sector through alteration in the slope or/and curvature of the yield curve. The joint interaction of the yield curve factors and the macro economy will be of immense importance to evaluate the impact of monetary and fiscal policies on the yield curve and the possible feedback effect on the real sector and foreign exchange market in the context of emerging markets. However, we focus on this issue in the future research. Overall, the estimated stochastic volatility pattern over time shows that the bond market in Pakistan is highly sensitive to the policy related moves and also to the economic track and fundamentals in the country. The market is also sensitive to external shocks that arise/happen in leading world markets (spillover effect from rest of the world). Furthermore, volatility is high during periods of SBP (State Bank of Pakistan) interventions and external (global) shocks (such as evident in the case of the world financial crisis of 2008). Moreover, the overall pattern of loadings against maturity is also roughly similar to that of Koopman *et al.* (2010) and Lips (2012) who find a remarkably lower sensitivity of the 1- and 9-year maturities. The difference in volatility pattern of two alternate specifications is also consistent with the results of Lips (2012), that EGARCH based framework yields higher volatility than GARCH specification.

As the three latent factors β_{1t}, β_{2t} and β_{3t} were categorized as level, slope and curvature factors respectively, here, we compare their estimates in all three frameworks with their empirical counterpart. The empirical counterparts level (*L*), slope (*S*) and curvature (*C*) are constructed from the observed zero-coupon yield data: (i) the level factor is defined as the 10-year yield (ii) the slope is the difference between the 10-year and 3-month zero rates, and (iii) the curvature as two times the two-year yield minus the sum of the 10-year and 3-

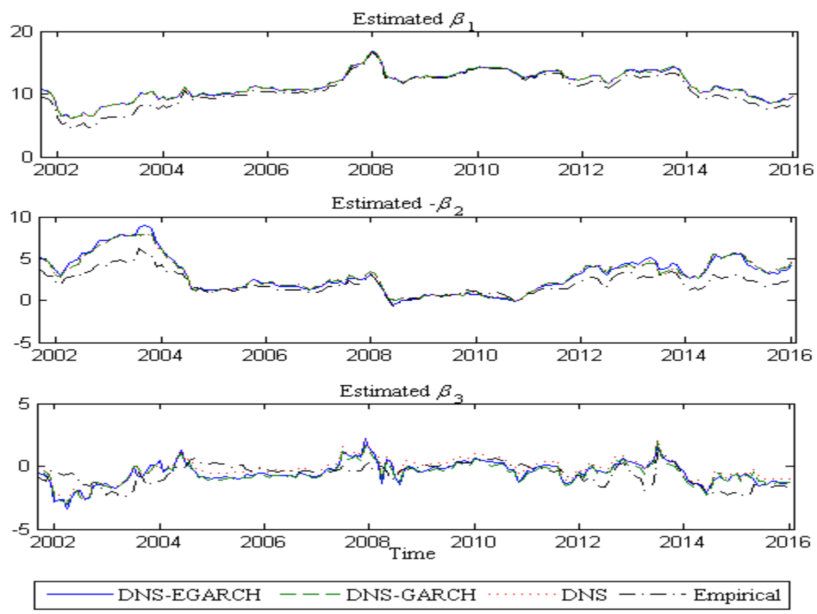
month zero coupon yields. The pairwise correlation of empirically defined level factor and $\hat{\beta}_{1t}$ (model based) is $\hat{\rho}\left(L_t, \hat{\beta}_{1t}^{DNS-EGARCH}\right) = 0.988$, $\hat{\rho}\left(L_t, \hat{\beta}_{1t}^{DNS-GARCH}\right) = 0.987$ and $\hat{\rho}\left(L_t, \hat{\beta}_{1t}^{DNS}\right) = 0.988$. The estimated pairwise correlation between the slope and $\hat{\beta}_{2t}$ is $\hat{\rho}\left(S_t, \hat{\beta}_{2t}^{DNS-EGARCH}\right) = -0.946, \hat{\rho}\left(S_t, \hat{\beta}_{2t}^{DNS-GARCH}\right) = -0.965$ and $\hat{\rho}\left(S_t, \hat{\beta}_{2t}^{DNS}\right) = -0.954$, while for the curvature (C) and $\hat{\beta}_{3t}$ is $\hat{\rho}\left(C_t, \hat{\beta}_{3t}^{DNS-EGARCH}\right) = 0.579$, $\hat{\rho}\left(C_t, \hat{\beta}_{3t}^{DNS-GARCH}\right) = 0.486$ and $\hat{\rho}\left(C_t, \hat{\beta}_{2t}^{DNS}\right) = 0.493$. The pairwise correlations for the DNS-EGARCH for all three factors is higher than that of the DNS-GARCH and DNS models. There is very small difference in estimated correlation for the $\hat{\beta}_{1t}$ and $\hat{\beta}_{2t}$ (not statistically significant), but somewhat larger difference for the $\hat{\beta}_{3t}$.

Overall, the analysis suggests that the estimated factors and the empirically defined factors follow the same pattern across time (shown in Figure 3), and therefore, $\hat{\beta}_{1t}$, $\hat{\beta}_{2t}$ and $\hat{\beta}_{3t}$ can be called level, slope and curvature factors, respectively.

To compare the errors of state equations across three models, we compute the covariance matrices of the transition innovations (denoted as Σ_v) and the results are presented in Table 3. It is evident from the results that all three diagonal elements of the matrix Σ_v that correspond to the variance of the state innovations are statistically significant, whereas only one off-diagonal element is statistically significant in all three setups.

Figure 3

Time Series Plot of Estimated Factors and Their Empirical Proxies



Model-based level, slope and curvature (i.e., estimated factors) are plotted against the data based level, slope and curvature (i.e., empirical proxies), where level is defined as the 10-year yield, slope as the difference between the 10-year and 3-month yields and curvature as two times the 2-year yield minus the sum of the 10-years and 3- month zero-coupon yields. Rescaling of estimated factors is based on Diebold and Li (2006).

Comparatively, in the time-varying volatility based frameworks most of the variance and covariance terms are much smaller than the counterpart terms in DNS setup. Moreover, the DNS-EGARCH comes up with a bit smaller variance and covariance terms than the DNS-GARCH. Furthermore, Wald test for the joint significance of the off-diagonal element in matrix Σ_v is employed, because most of the off-diagonal elements are statistically insignificant. The Wald test results suggest the joint significance of the covariance terms.

Table 3

Estimates of Covariance Matrix Σ

DNS-EGARCH			
	$q(.,1)$	$q(.,2)$	$q(.,3)$
$q(1,.)$	1.157 (0.186)		
$q(2,.)$	-0.001 (0.091)	0.458 (0.079)	
$q(3,.)$	0.206 (0.169)	0.180 (0.034)	0.427 (0.008)
DNS-GARCH			
$q(1,.)$	1.251 (0.192)		
$q(2,.)$	0.002 (0.001)	0.723 (0.001)	
$q(3,.)$	0.416 (0.397)	0.164 (0.478)	0.616 (0.005)
DNS			
$q(1,.)$	1.292 (0.093)		
$q(2,.)$	0.002 (0.014)	0.548 (0.108)	
$q(3,.)$	0.735 (0.622)	0.341 (0.125)	0.971 (0.063)

Note: The table shows the estimates of the covariance matrices of the innovations in the state equations for all three models, i.e., DNS-EGARCH, DNS-GARCH and DNS. The standard errors are in parenthesis. Bold entries denote that parameter estimates are significant at the 5% level.

To further evaluate the performance in terms of in-sample fitting, we present the summary statistics of the smoothed residuals of the observation equation in Table 4 for all three setups.

Table 4

Descriptive Statistics of the Yield Curve Residuals

Model	DNS-EGARCH Model			DNS-GARCH Model			DNS Model		
Maturity	MAE	RMSE	$\hat{\rho}\ (1)$	MAE	RMSE	$\hat{\rho}\ (1)$	MAE	RMSE	$\hat{\rho}\ (1)$
3	0.502	0.611	0.579	0.504	0.673	0.546	0.592	0.757	0.605
6	0.491	0.657	0.545	0.500	0.662	0.544	0.577	0.739	0.605
12	0.467	0.630	0.514	0.482	0.650	0.543	0.540	0.711	0.602
24	0.569	0.747	0.671	0.594	0.778	0.689	0.601	0.785	0.693
36	0.682	0.871	0.775	0.699	0.892	0.782	0.709	0.895	0.788
48	0.712	0.906	0.796	0.724	0.925	0.799	0.736	0.955	0.801
60	0.749	0.936	0.810	0.752	0.948	0.811	0.774	0.986	0.812
72	0.730	0.919	0.803	0.747	0.943	0.811	0.770	0.954	0.812
84	0.727	0.917	0.806	0.736	0.933	0.808	0.848	0.930	0.806
96	0.702	0.891	0.783	0.718	0.914	0.795	0.899	0.921	0.795
108	0.738	0.913	0.811	0.739	0.920	0.808	0.815	0.893	0.803
120	0.706	0.895	0.794	0.726	0.919	0.809	0.808	1.000	0.810

Note: The table presents summary statistic of the residuals for different maturity times of the measurement equation of DNS-EGARCH, DNS-GARCH and DNS models, using monthly data 2002:08–2016:12. RMSE and MAE is the root mean squared errors and mean absolute error respectively. $\hat{\rho}\ (i)$ denotes the sample autocorrelations at displacements of 1-month. The number of observations is 173.

The results show that the extended frameworks, *i.e.*, DNS-GARCH and DNS-EGARCH fit the observed yields more attractively as compared to the simple DNS model in terms of MAE as well as RMSE. Furthermore, the DNS-EGARCH and DNS-GARCH perform equally well to fit the yield curves in terms of MAE and residuals autocorrelation. However, DNS-EGARCH has a bit smaller RMSEs than the DNS-GARCH, but the difference may not be statistically significant. Overall, the results in Table 4 indicate that a more flexible and complex framework is required to fit the term structure of interest rates.

Summarizing, it turns out that the Nelson-Siegel model is capable of capturing the variation in yields across maturities and is closer to the true representation. Likewise, for developed economies, the Nelson-Siegel model is capable of distilling the term structure of interest rates quite well and describes the evolution and trends of the yield curve in emerging markets. Furthermore, the performance of volatility based extended models for the in-sample fit is consistent with the findings in Koopman *et al.* (2010) and Ullah *et al.* (2014) as they suggest that the addition of common volatility component in the standard DNS increases the flexibility to fit more complex shapes. Furthermore, our results provide evidence that besides fitting the yield curve a bit better, the EGARCH based specification is capable of capturing the asymmetric response of various yield to positive and negative shocks.

5. Term Structure Forecasting

Koopman *et al.* (2010) only report in-sample fit statistics for their DNS-GARCH model and no out-of-sample forecasting results. However, besides fitting current and describing past yield curve dynamics, term structure models are also used for what Diebold and Li (2006) call a key practical problem, namely to predict future interest rates. Hence it is useful to also evaluate the forecasting performance of the various models. The model, which comes up with a good approximation of the yield curve may not necessarily deliver attractive and satisfactory predictions of the future term structure of interest rates, therefore, in this section we assess the out-of-sample forecast performance of the standard DNS, DNS-GARCH and DNS-EGARCH models in comparison with the benchmark AR(1) model of yields.

The yield curve in our framework depends on the state vector β_t, therefore, yield curve forecasting is similar to forecasting yield curve factors. For forecasting the yield for various maturities, the yield curve latent factors are predicted with the help of the state equation and, then these predicted factors (state variables) are inserted in the signal equations (13) to compute the forecasted yields. In the first stage, we estimate each model over a subsample using the state space specification of (13-15) and in the next stage, predict the h-period ahead latent factors at each point of time in the out-of-sample forecast period by iterating forward the transition equation h-period ahead using the filtered state factors obtained in the previous stage. The h-period ahead predicted state vector is computed as:

$$\hat{\xi}_{t+h|t} = \left[I_s - \left(\sum_{i=0}^{h-1} \hat{F}^i\right)\right] \hat{C} + \hat{F}^h \hat{\xi}_{t|t} \qquad (23)$$

where: I_s is the (s × s) identity matrix ($s = 3$ and 4 for simple DNS, and volatility based extended models respectively), $\hat{C}$ and $\hat{F}$ are the parameters estimates of the state equation and $\hat{\xi}_{t|t}$ is the most recent available estimated factors vector in the update step.

Furthermore, the estimation and forecasting is made recursively, using data from 2002:08 to the time that the forecast is made, beginning in 2012:01 and extending through 2016:12

for $h = 1{,}6$ and 12 (*i.e.,* three distinct forecast horizons are considered). Subsequently, the forecasted the state vector in (13), the h-month ahead forecast is computed as:

$$\hat{R}_{t+h|t} = \hat{B}\hat{\xi}_{t+h|t} \tag{24}$$

where: $\hat{R}_{t+h|t}$ is the forecast of the yield at time t for $t+h$ period (denoted as $\hat{R}_{t,t+h}$), and $\hat{\xi}_{t+h|t}$ is the h-period ahead predicted vector of state variables. In the next stage, the forecast errors at $t+h$ are computed as $e_{t,t+h} = R_{t+h} - \hat{R}_{t,t+h}$, where R_{t+h} is the actual observed yield vector at $t+h$ and the $\hat{R}_{t,t+h}$ is the h-month ahead forecasted yields in period t.

Furthermore, the AR(1) model of yields for forecasting the term structure of interest rate serves as a benchmark for the comparison. The AR specification to compute the forecast of yields for various maturities is:

$$R_{t+h}(m) = \delta_0 + \delta_1 R_t(m) + \varepsilon_{t+h} \tag{25}$$

for $h = 1, 6,$ and 12, and $\varepsilon_t \backsim N(0, \sigma^2)$.[5]

5.1. Term Structure Forecast Results

Using the four specification to forecast the future path of yields for all 15 maturities (considered in this study), *i.e.,* the DNS-EGARCH, DNS-GARCH, DNS and the AR(1) models for each forecast horizon, we calculate the forecast errors and compute their descriptive feature, such as mean absolute error (MAE), root mean squared errors (RMSE) and autocorrelation at various displacements. Table 5 presents the results of the forecasts of all four specifications for maturities of 3, 6, 12, 24, 30, 36, 60, 96 and 120 months of the $h = 1{,}6$ and 12 months. The table reports three main aspects, MAE, RMSE and errors persistency of forecast errors to compare the out-of-sample forecasts performance of each model.

The results of one month ahead forecast are presented in table 5. All three specifications of the Nelson-Siegel model outperform the AR(1) yield forecasts in terms of all descriptive feature of the forecast errors, whereas, the standard DNS performs slightly better than the volatility based extended models, *i.e.,* DNS-GARCH and DNS-EGARCH models. The MAE and RMSE for most of the maturities of the DNS are a bit smaller than the rest of the two models, however, in terms of the errors autocorrelation, the volatility based extended models outperform the DNS. Moreover, the two volatility based models (DNS-GARCH and DNS-EGARCH) have almost similar performance in terms of MAE, RMSE and errors persistency.

The results in Table 5 for the 6 months and one year ahead forecast show that the forecast errors become larger as we lengthen the forecast horizon. Similar to the one-month ahead forecast, all three Nelson-Siegel type models outpace the benchmark AR(1) forecasts in all three descriptive features for $h = 6,$ and 12. Among the remaining three models, the order of superiority runs from DNS-EGARCH to DNS-GARCH and to DNS. For $h = 6$ and 12 months, the MAE and RMSE for the DNS-EGARCH is reasonably smaller for all maturities than the rest of the two specifications of yield, however, the errors persistency is similar in all three setups. Moreover, the DNS-EGARCH forecasts beat the corresponding DNS forecasts in terms of MAE, RMSE as well as errors autocorrelation for both 6 and 12 months ahead forecast horizons. However, in regard of MAE and RMSE,

[5] *Other time series specifications, such as random walk model can also be a benchmark model. However AR(1) specification of yield outpaces forecasts from this model. Hence we do not find it necessary to report results from this model.*

the improvement of DNS-EGARCH over DNS-GARCH is very minor in both 6 and 12 months ahead forecasts.

In summary, the out-of-sample forecast results of the three Nelson-Siegel specifications seem reasonably accurate in terms of lower forecast errors than the benchmark AR(1) model of yield. Moreover, the volatility based extended specifications have better performance than the DNS at 6- and 12-month ahead forecast horizons, while the latter comes with much accurate forecasts for the short horizon forecasts, i.e., one month, than the DNS-GARCH and DNS-EGARCH models. Furthermore, the forecast errors of the EGARCH based model are almost identical to the GARCH based setup for $h = 1$, while a bit smaller for 6- and 12-month ahead forecast horizons. The serial correlation of forecast errors in the DNS may likely be from the pricing errors and illiquidity. Overall, the out-of-sample forecasting results are similar with the one presented in Lips (2012), who finds that allowing for time-varying volatility in the DNS enables the model to better capture dynamics in the most volatile yields and produce relatively accurate 6- and 12-month ahead forecasts.

Table 5

Out-of-sample Forecasting Results

Maturity	DNS-EGARCH			DNS-GARCH			DNS			AR(1)		
	MAE	RMSE	$\hat{\rho}$ (1)	MAE	RMSE	$\hat{\rho}$ (1)	MAE	RMSE	$\hat{\rho}$ (1)	MAE	RMSE	$\hat{\rho}$ (1)
1 month ahead forecasting												
3	0.536	0.622	0.577	0.500	0.618	0.523	0.419	0.456	0.850	1.072	1.274	0.937
12	0.492	0.529	0.513	0.427	0.500	0.482	0.422	0.487	0.945	1.117	1.342	0.944
24	0.648	0.764	0.659	0.635	0.750	0.639	0.611	0.717	0.988	1.140	1.405	0.949
36	0.849	0.955	0.791	0.857	0.977	0.810	0.802	0.928	0.809	1.160	1.394	0.950
60	0.899	0.988	0.979	0.907	0.984	0.820	0.855	0.967	0.806	1.169	1.385	0.954
96	0.842	0.945	0.783	0.840	0.964	0.794	0.816	0.935	0.799	1.191	1.419	0.951
120	0.769	0.891	0.775	0.787	0.921	0.788	0.766	0.895	0.792	1.011	1.253	0.950
6 months ahead forecasting												
3	1.087	1.259	0.839	1.125	1.306	0.756	1.754	2.061	0.780	2.111	2.300	0.937
12	1.125	1.313	0.847	1.086	1.276	0.783	1.612	1.923	0.816	2.175	2.373	0.948
24	1.373	1.333	0.887	1.402	1.742	0.858	1.847	2.280	0.880	2.217	2.422	0.942
36	1.582	1.923	0.911	1.617	1.952	0.899	1.995	2.447	0.906	2.242	2.432	0.944
60	1.734	2.028	0.917	1.757	2.069	0.913	2.054	2.465	0.917	2.309	2.500	0.959
96	1.749	2.101	0.929	1.789	2.108	0.925	2.035	2.408	0.926	2.290	2.505	0.953
120	1.745	2.102	0.928	1.783	2.105	0.926	1.974	2.358	0.925	3.127	2.355	0.954
12 months ahead forecasting												
3	1.077	2.145	0.909	1.978	2.394	0.861	1.927	2.506	0.881	3.128	3.292	0.937
12	1.768	2.150	0.916	1.820	2.153	0.872	1.942	2.268	0.906	3.218	3.394	0.944
24	1.732	2.018	0.934	1.794	2.025	0.905	1.852	2.304	0.935	3.225	3.416	0.931
36	1.617	1.758	0.950	1.682	1.929	0.935	1.977	2.463	0.952	3.227	3.411	0.936
60	1.446	1.766	0.955	1.463	1.768	0.945	1.974	2.447	0.956	3.266	3.456	0.946
96	1.330	1.666	0.957	1.374	1.707	0.950	1.955	2.400	0.958	3.246	3.457	0.936
120	1.337	1.668	0.960	1.390	1.712	0.955	1.887	2.334	0.961	3.156	3.380	0.948

Note: The table reports the results of out-of-sample forecasting using state-space specification for the DNS-EGARCH, DNS-GARCH and DNS models along with the AR(1) forecasts of yields for various maturities. We estimate the models recursively from 2002:08 to the time that the forecast is made. We define forecast errors at $t + i$ as $R_{t+i}(m) - \hat{R}_{t,t+i}(m)$, where $\hat{R}_{t,t+i}(m)$ is the $t + i$ month ahead forecasted yield at period t, and we report the mean absolute errors (MAE) and root mean squared errors (RMSE) of the forecast errors, as well as their first order sample autocorrelation coefficients.

5.2. Out-of-sample Forecast Accuracy Comparisons

While the results in the previous section show clear differences between the forecast accuracy of the four specifications, it is essential to evaluate the statistical significance of these differences. Here, we employ the Diebold and Mariano (1995) test that provides a means to compare the mean square errors of the two competing forecast errors. The Diebold and Mariano (DM) test is a standard statistical test that compares the squared forecast errors of two competing models and is the most commonly applied test for comparing the forecast accuracy.

However, the focus in this study is on comparing the three Nelson-Siegel (1987) base specifications, *i.e.*, DNS-EGARCH, DNS-GARCH and DNS model; we make the comparison among the four specifications of yield forecasting in four different pairs. Based on the results in the previous section, in the first pair, DNS is compared against the AR(1) specification of the yield, while the forecast errors of DNS-EGARCH and DNS-GARCH are compared with the standard DNS forecast errors in the second and third pair respectively. The fourth pair draws comparison between the DNS-EGARCH and DNS-GARCH.

The DM-statistics for each pair of models and each forecast horizon of the Diebold–Mariano test are provided in Table 6 that reflects the differences in the RMSE shown in Table 5. It is worthwhile to mention that besides the statistical significance; the sign of the statistic has an important interpretation in the context of DM-test. In our framework, the negative sign indicates the superiority of the model mentioned first in the first column of the table, such as in the first pair the negative sign show the preference of DNS over AR(1) model and vice versa.[6] The results in Table 6 for the first pair show a universally significant difference in the squared errors for all three forecast horizons and all maturities of the standard Nelson-Siegel model (DNS) and AR(1) model, as all DM statistics are statistically significant. Moreover, all test-statistics are negative indicating that the DNS model provides more accurate forecasts for all maturities than the benchmark AR(1) model.

The DM-statistics (for the second and third pairs DNS against DNS-GARCH and DNS-EGARCH models, respectively) reported in Table 6 indicate statistically significant difference of the RMSE for the one-month-ahead forecast for most of the maturities. The p-value is greater than 0.1 for only two maturities (30- and 36-month maturities in case of DNS-GARCH) for $h = 1$, while the difference of the RMSE between the two models is statistically different from zero for the remaining maturities. Whereas in the case of DNS-EGARCH all test statistics are positive and significant. It indicates that the DNS outpaces unanimously the DNS-GARCH as well as DNS-EGARCH for the very short horizon forecasts. Here the positive sign points towards the superiority of DNS over the competing model. Comparing the 6- month ahead forecasts of DNS in the second and third pair, the results point towards the universal preference of volatility based extended models over standard DNS as most test-statistics are negative and statistically significant. For $h = 12$, the DNS-EGARCH comes up with more accurate forecasts than the DNS (most test-statistics are negative and significant), while the forecast errors of DNS and DNS-GARCH are almost identical in statistical terms.

[6] *In carrying out the DM-test, we compute the difference between the squared errors of the two competing models as: $d_t = e_{1t}^2 - e_{2t}^2$, where e_{1t}^2 is the squared forecast errors of DNS and e_{2t}^2 is the squared forecast errors of AR(1) model. Since, the negative value of d_t indicates that the DNS have lowered squared forecast errors as compared to the AR(1) specification of yield.*

Table 6

Diebold-Mariano Test-statistic

Maturity	DNS against the AR(1)			DNS-EGARCH against the DNS		
	$h = 1$	$h = 6$	$h = 12$	$h = 1$	$h = 6$	$h = 12$
3	-2.931**	-3.588***	-2.219**	3.411***	-2.943***	-1.220
12	-3.349***	-2.945***	-2.874***	4.835***	-2.428**	-0.255
24	-1.987**	-5.669***	-3.314***	3.389***	-2.365**	-0.397
36	-2.198*	-2.768**	-2.373**	1.272	-2.329**	-0.798
60	-2.188*	-1.997*	-3.482***	3.186***	-2.533**	-1.255
84	-4.589***	-2.005*	-3.492***	8.154***	-2.859***	-1.141
96	-3.952***	-2.832***	-3.492***	4.835***	-2.855***	-1.747*
120	-3.427***	-3.891***	-3.441***	4.621***	-2.831***	-1.802
	DNS-EGARCH against the DNS			DNS-EGARCH against the DNS-GARCH		
3	7.092***	-2.966***	-2.375**	0.765	0.839	-2.678**
12	5.512***	-2.408**	-2.386**	-0.673	0.549	-2.161**
24	5.097***	-2.384**	-3.340***	-3.374***	-0.292	-1.105
36	3.226***	-2.427**	-2.735***	-2.615**	0.175	-1.773*
60	2.153**	-2.657**	-1.223	-3.762***	-0.250	-0.126
84	3.971***	-3.013***	-1.636*	-4.348***	-0.898	-1.799*
96	2.393**	-2.960***	-1.764*	-3.221***	-0.148	-1.832*
120	1.998**	-2.879***	-1.818*	-2.863***	-0.058	-1.739*

*Note: The table presents Diebold–Mariano forecast accuracy comparison test results for the two different pairs of models, i.e., the DNS-EGARCH, DNS-GARCH, DNS models and AR(1) forecasts for 1, 6 and 12 months ahead forecasts. The null hypothesis is that the two forecasts have the same root mean squared error. ***, ** and * denote that the test statistic is significant at the 1%, 5% and 10% level respectively.*

Among the two volatility based extended models in the fourth pair, the results indicate that for $h = 1$ the DNS-EGARCH has preference over DNS-GARCH for long and medium term maturities, whereas both perform equally well for the short spectrum of maturities. In the case of 6- month ahead forecast horizons, both models have similar performance as all test-statistics are statistically not different from zero except the 108-month maturity. For the 12-month ahead horizon, the results show that 9 out of 15 DM-statistics show statistically significant (at the 10% significance level) superiority of the DNS-EGARCH model over the DNS-GARCH model.

The results of the DM test show that the Nelson–Siegel based yield curve specifications outperform competing benchmark forecast models such as the AR(1). Moreover, within the class of Nelson–Siegel models, the volatility based extended models have greater accuracy and success than the DNS model in forecasting yields for medium- and long-forecast horizons, while the simple DNS has more attractive performance the one month ahead horizon.

6. Conclusion

The term structure of interest rates is considered as an important indicator of the economy as well as capital markets. It is considered a highly reliable source for contingent claims pricing, determining the cost of capital and managing financial risk. It is also widely used for understanding investors' sentiments about the future conditions in the economy. In this study, we consider the standard Nelson-Siegel model and its extended versions that take into account the time varying volatility, *i.e.,* DNS-GARCH and DNS-EGARCH and compare them in terms of in-sample fitting as well as the out-of-sample forecast performance. We use monthly Pakistani government bonds zero coupon data (yield to maturity) from August 2002 until December 2016 to carry out the empirical analysis. The study contributes to literature by expanding the earlier work of Koopman, et al. (2010). Besides looking at more elaborate specifications for volatility modeling such as E-GARCH, it also evaluates the predictive role of time-varying volitlity in out-of-sample forecasting.

For the in-sample fit, the results show that the Nelson-Siegel model is capable of distilling the term structure of interest rates quite well and describes the evolution and trends of the yield curve in emerging markets as well as in the context of larger and developed markets. However, the magnitude of error in emerging markets is reasonably larger as compared to the developed markets. This might be due to ignoring the arbitrage free restriction or pricing error in the market (possibly because of lack of liquidity). Furthermore, the volatility based extended models, *i.e.,* DNS-GARCH and DNS-EGARCH, are capable of fitting the yield curve more accurately than the standard DNS model, particularly in periods of high volatility. Moreover, the DNS-EGARCH model fits the curve slightly better than the DNS-GARCH model.

Regarding term structure forecasting, we conclude that all three specifications of yield curves based on the Nelson-Siegel functional form can replicate the interest rates' general trends in emerging economies. The out-of-sample forecast results of the Nelson-Siegel specifications seem reasonably accurate in terms of low forecast errors and outperform the benchmark time series forecast models of yields, such as AR(1) and random walk models. Moreover, by allowing for time-varying volatility in the model (DNS-EGARCH and DNS-GARCH), the term structure model better captures the dynamics in most volatile yields. It also tends to produce more accurate forecasts at both 6- and 12-month ahead horizons. However, the forecast errors of the simple DNS model in terms of RMSE are reasonably smaller as compared to the volatility based extended models at the short one-month forecast horizon. We may conclude that DNS-EGARCH model has excellent performance for medium and longer forecast horizons. It also turns out that the richer parameterization of the model leads to a better in-sample fit and out-of-sample performance.

The extensions on the work of Koopman, *et al.* (2010) presented in this study offers several directions for further research. The results show that the interest rate volatility dynamics might be captured more efficiently by an asymmetric model, however an alternate specification such as GARCH-X including macroeconomic factors may also be helpful to capture volatility more efficiently. Secondly, rather than only considering the common volatility component in the observation equation, a similar specification can also be included in the state equation to capture the volatility in the 3-factors of yields.

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Appendix-I: Coefficients and Latent Variable in the General State-space Form

In the statistical formulation of the models in section 2.3, the matrices and vectors for the state and observations equations should be considered as follows. The matrices and vectors in state-space system in (13-15) for the simple DNS model should be defined as:

$B = \Lambda(\lambda)$: (N×3) $\quad \xi_t = [\beta_{1t}, \beta_{2t}, \beta_{3t}]'$: (3×1) $\quad w_t = \varepsilon_t$: (N×1)

$C = [I_3 - A]\mu$: (3×1) $\quad F = A$: (3×3) $\quad u_t = v_t$: (3×1)

$\Omega = \Omega$: (N×N) $\quad Q_t = \Sigma_v$: (3×3)

while, for the DNS-GARCH and DNS-EGARCH models in the state-space system presented in (10-12), can be written as:

$B = [\Lambda(\tau) \quad \Gamma_\varepsilon]$: (N×4) $\quad \xi_t = \alpha_t = [\beta_{1t}, \beta_{2t}, \beta_{3t}, \varepsilon_t^*]'$: (4×1)

$C = \begin{bmatrix} (I_3 - A)\mu \\ 0 \end{bmatrix}$: (4×1) $\quad F = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$: (4×4)

$u_t = \begin{bmatrix} v_{t+1} \\ \varepsilon_{t+1}^* \end{bmatrix}$: (4×1) $\quad Q_t = \begin{bmatrix} \Sigma_v & 0 \\ 0 & h_{t+1} \end{bmatrix}$: (4×4)

$w_t = \varepsilon_t^+$: (N×1) $\quad \Omega = \Omega$: (N×N)

In all three specifications $\Lambda(\lambda)$ is (N×3) matrix of loadings, Γ_ε is (N×1) vector showing the sensitivity of various yields to common volatility component, $\beta_t = (\beta_{1t}, \beta_{2t}, \beta_{3t})'$ is the (3×1) vector of latent factors of the yield curve, $\alpha_t = (\beta_t', \varepsilon_t^*)'$ is the (4×1) vector of latent factors, μ is (3×1) vectors of factors mean, and A is (3×3) full-matrices of parameters. The ε_t and v_t are (N×1) and (3×1) innovations vectors of the observation and state equations respectively, Ω is (N×N) covariance matrix of the measurement equation innovations, and Σ_v is (3×3) lower triangular covariance matrix of the state innovations. Moreover, ε_t^+ is (N×1) errors vector of the observation equation in two time time-varying volatility (GARCH and EGARCH) based models.